Ph.D. Thesis

ABSTRACT

EFFICIENCY AND ERROR IN THE FINITE ELEMENT METHOD WITH APPLICATION TO STRUCTURES

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I. INTRODUCTION

The Finite Element Method (FEM) is a numerical method used to find the approximate solution of the differential equations governing, for instance, the behavior of structures. The survey of errors in the solution and the adaptivity of FEM is one of the main concerns of a part of the scientific community, in the period 1990-2000 being published 2177 papers and theses studying the problem of errors.

The accuracy of the results obtained using FEM is usually evaluated through some error indicators. If the error in a part of the model is higher than some bounds, the mesh should be refined on that part of the model. The refinement procedures used are: h, p, hp and r. No matter the procedure used, the process is the same: the problem is solved, the error is estimated, if needed, the mesh is modified.

Starting with 1990 a new meshing technique appeared. Based on the idea of genetic programming, a group of researchers studied the possibility of solving the triangulation problem with genetic algorithms. The results obtained so far are presented in the thesis; the number of the papers published is small, showing that the integration of meshing issues and evolutionary programming is at its initial stages.

To get closer to the intended purpose (an evolutionary adaptive meshing algorithm integrated with a FEA computer program), I studied the results provided by an evolutionary meshing procedure on a set of initial triangles. As mentioned, the number of publications in the field is very small and, from my knowledge no other authors have investigated the problem of refining an initial triangulation.

The program was implemented by my husband, Dan-Ioan Danciu, master student at the Faculty of Automation and Computer Science from Cluj-Napoca; the program allows the refining of an initial bidimensional triangulation. The refinement starts from the contour of the body but it can have as a starting point several points of interest. When the program will be integrated with FEA software, the points of interest will be the elements error indicators.

For the bidimensional FE I studied several criteria used to give information about the element distortion and I proposed a set of new criteria. The criteria for the triangle elements have been implemented and tested in the proposed evolutionary procedure.

II. PHYSICAL AND MATHEMATICAL PRELIMINARIES

The second chapter presents several physical and mathematical elements used in the thesis: the classical plate theory (Kirchhoff plate theory) and the first order shear deformation theory (Reissner-Mindlin theory); partial differential equations used in the FEM formulation and in the study of errors in the FEM; vector norms, the divergence theory and and operations with the operator.

III. FEM FORMULATION

The third chapter presents the variational formulation of the Finite Element Method. The essence of the formulation resides in the calculation of the total potential energy of the system . The total potential energy is called the functional of the problem. Next the Galerkin method is presented and a parallel between the Galerkin method and the variational formulation is made.
Chapter four starts with a classification of the finite elements, followed by an overview of the requirements conformal FEs should fulfill. The numerical integration rules for 1D, 2D and 3D cases are presented. The bidimensional elements, quad and triangle are presented in detail with an emphasis on the distortion measures. For the triangle element with 3 nodes a new distortion measure, called a fitness function of the triangle, is introduced:

\[ f_{\text{tri}} = \frac{D}{d} = \frac{2abc}{(-a+b+c)(a-b+c)(a+b-c)} \]  

Where \( D \) is the diameter of the circumcircle and \( d \) is the diameter of the incircle and \( a, b \) and \( c \) are the sides of the triangle. The proposed fitness function is used in the sixth chapter for the sanity check of the new formed triangles.

For the quad elements with 8 nodes I proposed a set of four new distortion measured named fitness functions:

\[ F_{d1} = \frac{\max(d_{1-3}, d_{2-4})}{\min(d_{1-3}, d_{2-4})} = \frac{\max(\sqrt{(\alpha_1 + \alpha_2)^2 + (\beta_1 + \beta_2)^2}, \sqrt{(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2})}{\min(\sqrt{(\alpha_1 + \alpha_2)^2 + (\beta_1 + \beta_2)^2}, \sqrt{(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2})} \]

\[ F_{d2} = \frac{\max(d_{5-7}, d_{6-8})}{\min(d_{5-7}, d_{6-8})} = \frac{\max(\sqrt{\alpha_5^2 + \beta_5^2}, \sqrt{\alpha_6^2 + \beta_6^2})}{\min(\sqrt{\alpha_5^2 + \beta_5^2}, \sqrt{\alpha_6^2 + \beta_6^2})} \]

\[ F_{d3} = \frac{\max(d_{5-1}, d_{3-7})}{\min(d_{5-1}, d_{3-7})} \]

\[ F_{d41} = \frac{\max(d_{m-6}, d_{m-8})}{\min(d_{m-6}, d_{m-8})}, \quad F_{d42} = \frac{\max(d_{m-7}, d_{m-5})}{\min(d_{m-7}, d_{m-5})} \]

The performance of the fitness functions have been evaluated on two sets of problems: a cantilever loaded by a force at its free end modeled using QUAD8 FEs and a simply supported square plate loaded by a uniformly distributed force modeled using quadrilateral 8 node elements with 6 degrees of freedom, formulated after the Mindlin plate theory. In the numerical simulation different types of element distortion have been used.

V. ERROR ESTIMATION

The fifth chapter addresses the problem of error estimation in the FEM. An overview of the error estimation techniques is presented with an emphasis on the a posteriori error estimation techniques. For the residual methods the global explicit residual error estimators and the element implicit error estimators are presented. The recovery procedure proposed by Zienkiewicz and Zhu and some numerical examples are presented.

Using the element residual error estimators and two adaptive strategies, a maximum and an equilibrium strategy, I conducted numerical tests to investigate the performance of the error estimators combined with the adaptive strategies. The model problem used was Poisson’s equation with unit load on a square and on an L-shaped domain with different initial mesh.
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configurations. The number of the new FEs generated was investigated, as well as the computational time needed for the estimated error to be less than 1%.

VI. EVOLUTIONARY METHOD FOR RE-TRIANGULATION

The sixth chapter presents the evolutionary procedure proposed through this thesis. In the first part an overview of the use of evolutionary methods in meshing is done.

The purpose of the present chapter is to show that, by using evolutionary procedures, after a few generations, one can obtain acceptable triangular meshes that could be used in analysis after a minimum correction. The triangulations obtained are optimized after two parameters: the quality of the triangles obtained and the refinement of the mesh. Through the implemented operators I tried to keep the quality of the newly generated triangles and, where possible, to improve the quality of the initial triangles.

All the steps of the procedure are presented in detail. The operators presented in the literature have been implemented and their performance evaluated for the new procedure proposed. Two new cross-over operators have been proposed, explained in detail and implemented, as well as a new mutation operator responsible for the mesh refinement. A situation appearing during mutation was named as fan mutation and ways to resolve it have been presented. The fitness of the individual is evaluated through a cumulative function, that takes into account the quality of the individual through the deviation fitness \( f_d \) and the refinement obtained through the refinement fitness \( f_r \).

\[
f_d = \frac{3\max(f_{d,t}) + 4 \cdot \text{avg}(f_{d,t}) + \min(f_{d,t})}{8}
\]

\[
f_r = \frac{\text{count}(T)}{n_{\text{ideal}}} \cdot \frac{A(T)}{\sqrt{3} f_{\min}^2}
\]

\[
f = c_d \cdot f_d + c_r \cdot f_r
\]

Where \( c_d = 0.55 \) and \( c_r = 0.45 \).

For a plane octagonal domain the influence of the number of generations, the division rate, the mutation rate, the cross-over rate and the initial triangulation are investigated.

Next the issue of solving a domain with holes is addressed. The procedure starts with a set of initial triangles that take into account the geometry of the body to be meshed and the operators proposed are not modifying the geometry; therefore, any type of geometry can be meshed.

The strong and the weak points of the procedure are investigated and ways of improving it are proposed at the end of the chapter. As well, a short overview of the unstructured meshing techniques is done.

Is is shown in this chapter that the proposed procedure can improve the fitness of the initial triangulation with 25 to 56.5%. The number of generations for which the results were obtained is relatively small, 25 to 100 generations. It was shown that, for the operators implemented, the initial triangulation can affect the final result, the evolution of those individuals having a common vertices for all the triangles is much weaker than the evolution of the other individuals. It was shown that the increase of the number of individuals in the population and by increasing the number of generations, the results can be improved. Therefore one of the recommendations is the investigation of the performances of the procedure in grid or cloud computing.
VII. CONCLUSIONS, CONTRIBUTIONS AND FUTURE WORK

The seventh chapter briefly presents the Ph.D. Thesis content, principal original contributions and future research directions.

The main original contributions of this Ph.D. thesis are:

1. The proposal of five new distortion measures for triangle and quad finite elements;
2. A documented synthesis of the error estimation techniques and numerical studies for two types of residual estimators and two adaptivity strategies;
3. A comprehensive review of the use of genetic algorithms in meshing problems;
4. The introduction of a new meshing procedure, based on an evolutionary strategy;
5. A comprehensive review of the literature on evolutionary methods and triangulations;
6. The proposal of two new cross-over operators: a reduced crossover operator with validation and a full crossover operator with learning;
7. The proposal of a new mutation operator: the division operator;
8. For the flip type mutation operator, the identification of a special case named fan mutation and a proposal of how to solve the new mutation;
9. The proposal of the fitness function:
   - A perfection fitness function used for the sanity check;
   - A deviation fitness function used to evaluate the quality of an element of the individual (a triangle) and a fitness function used to evaluate the quality of the whole individual;
   - A refinement fitness function used to assess the quality of the mesh refinement;
   - The cumulative function;
10. The assessment of the performance of the proposed procedure when the parameters of the procedure are varied: the mutation rate, the division rate, the crossover rate, the number of generations, the influence of the initial individual over the final solution.

SELECTIVE REFERENCES

Cap. I.

Cap. II.
Cap. III

Cap. IV

Cap. V
Efficiency and error in the Finite Element Method with application to structures

Cap. VI.